Digital Communication Systems
ECS 452

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8. Optimal Detection for Additive Noise Channels

1-D Case

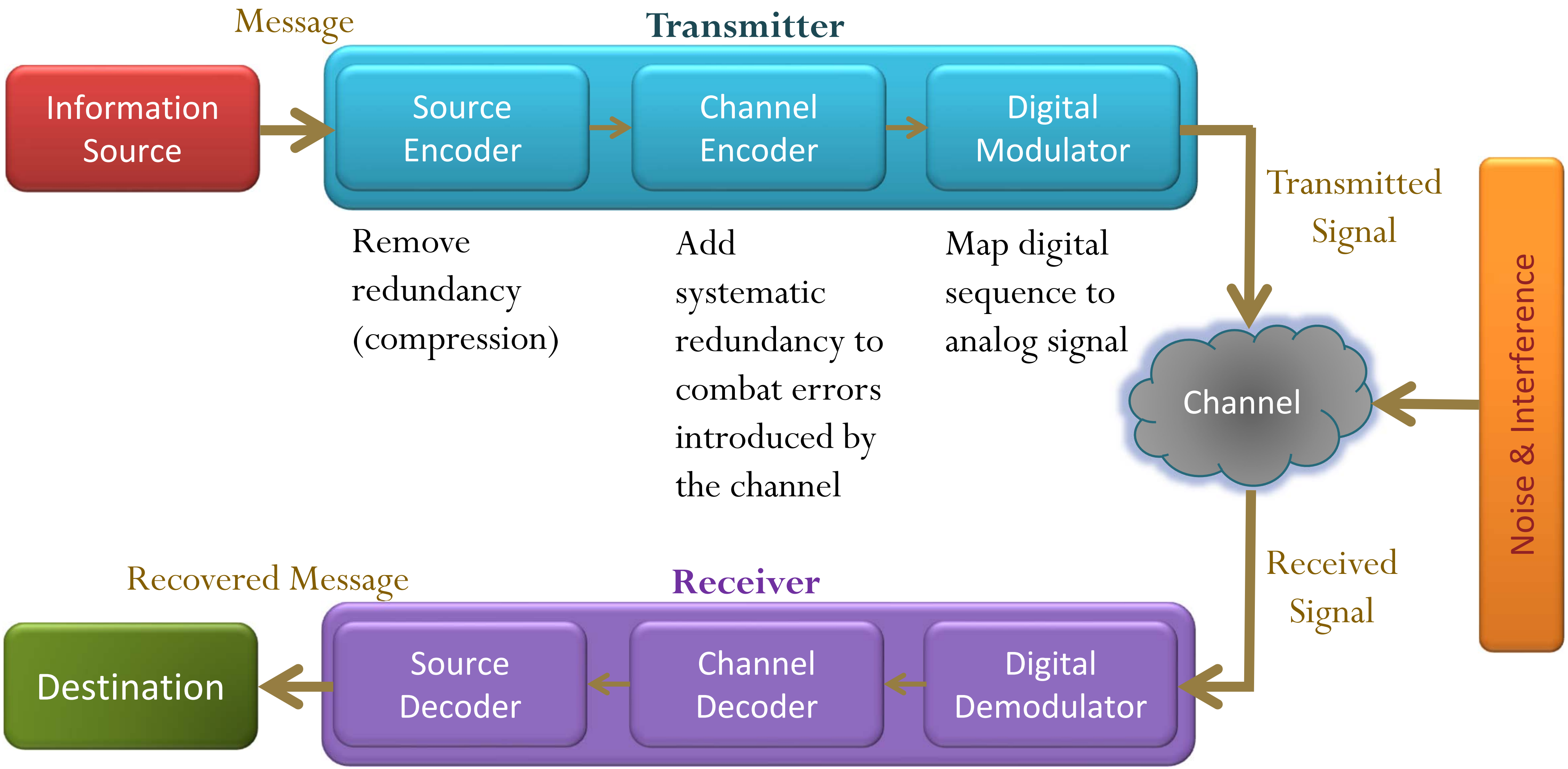
Office Hours:

Check Google Calendar on the course website.

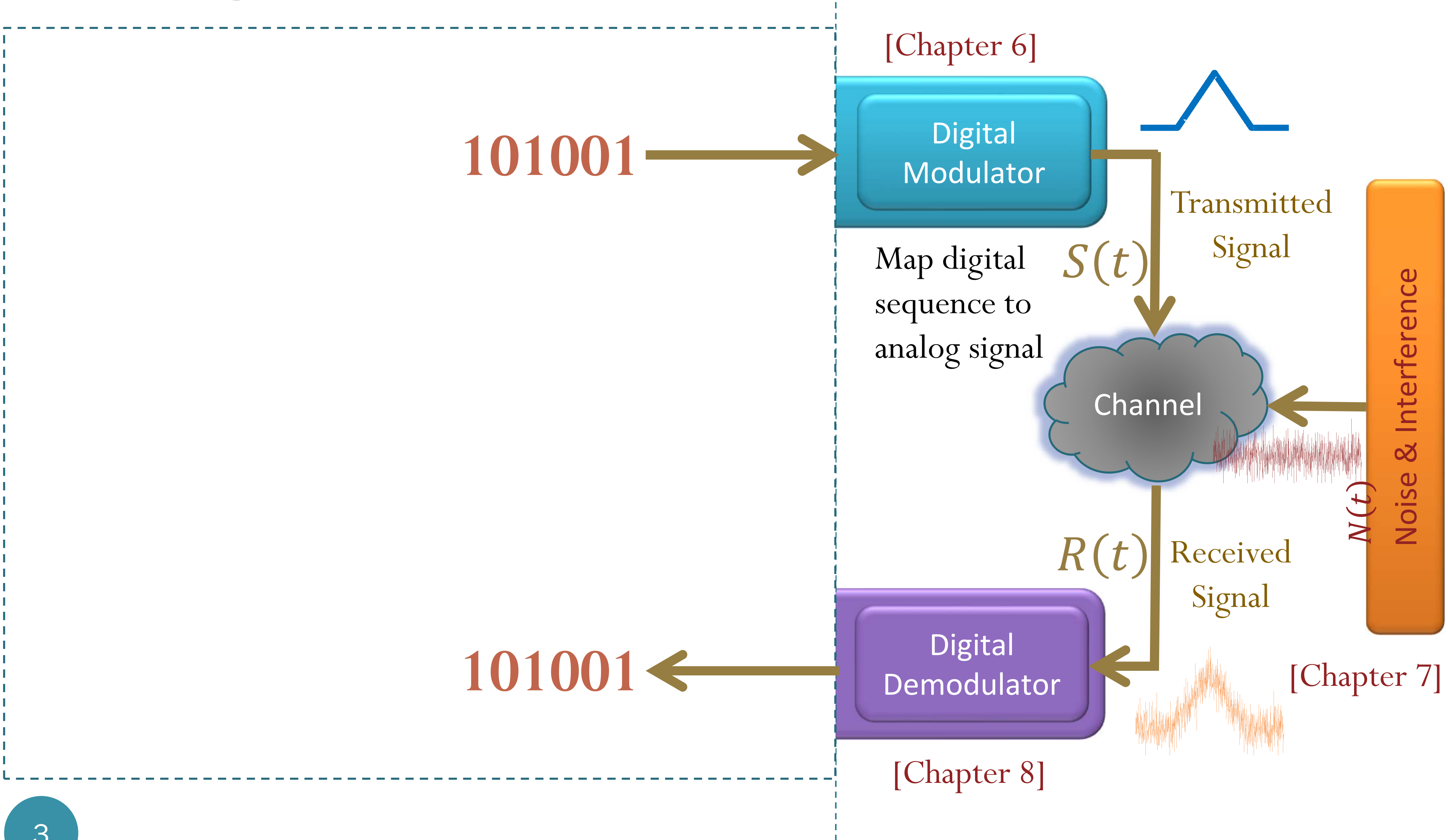
Dr.Prapun's Office:

6th floor of Sirindhralai building, BKD

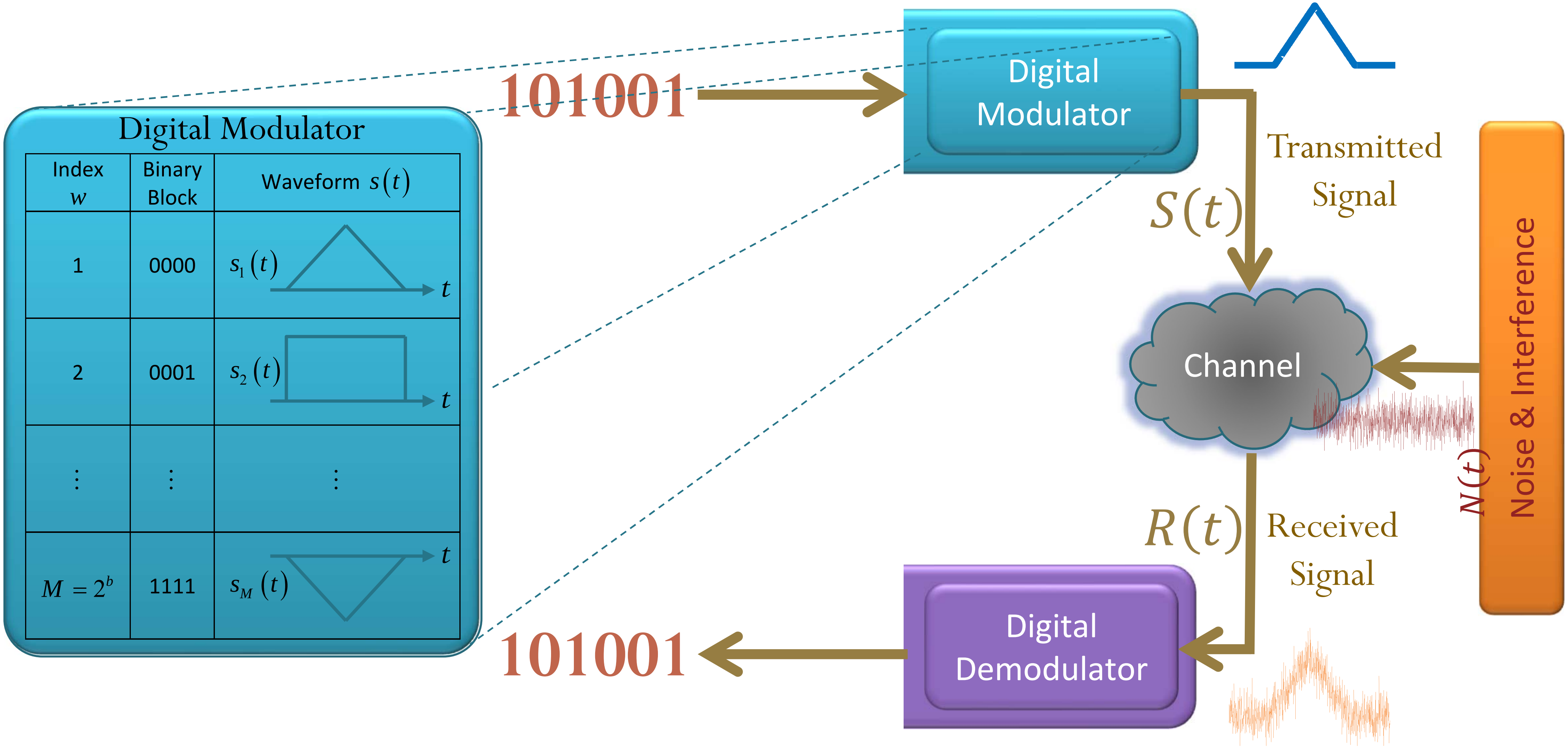
Elements of digital commu. sys.



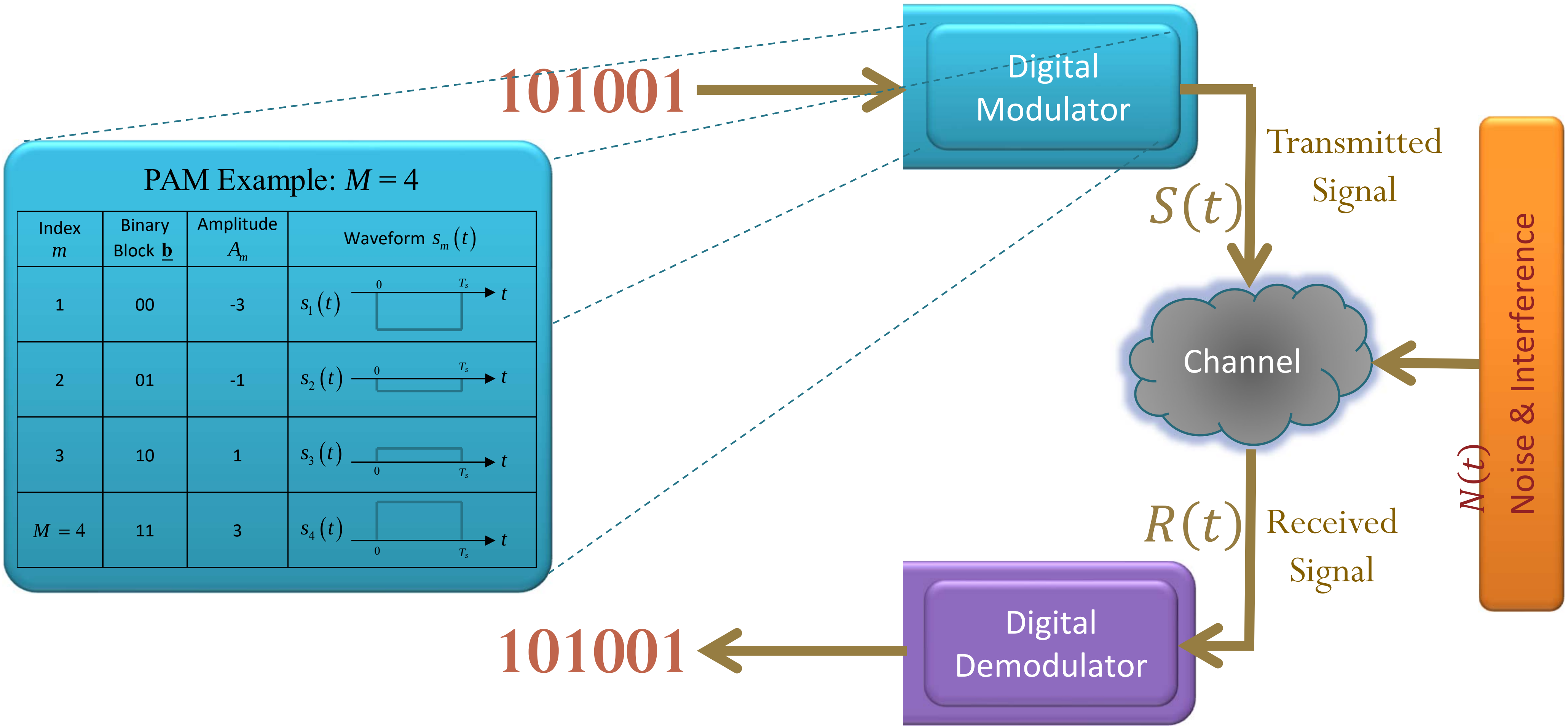
Digital Modulation/Demodulation



Digital Modem: Ex 1

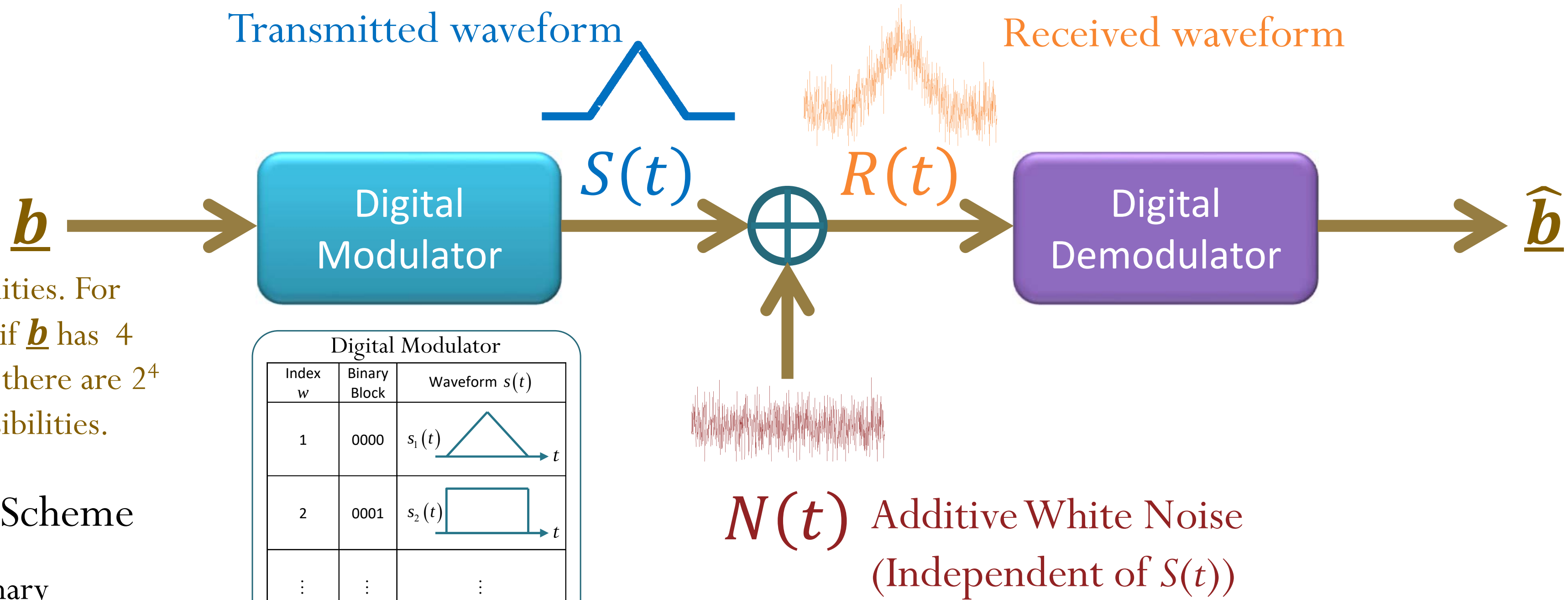


Digital Modem: Ex 2



Index m	Binary Block $\underline{\mathbf{b}}$	Amplitude A_m	Waveform $s_m(t)$
1	00	-3	$s_1(t)$
2	01	-1	$s_2(t)$
3	10	1	$s_3(t)$
$M = 4$	11	3	$s_4(t)$

Analysis of Digital Modem



M possibilities. For example, if b has 4 bits, then there are $2^4 = 16$ possibilities.

Digital Modulator

Index w	Binary Block	Waveform $s(t)$
1	0000	$s_1(t)$
2	0001	$s_2(t)$
⋮	⋮	⋮
$M = 2^b$	1111	$s_M(t)$

***M*-ary** Scheme

M =2: Binary
M = 3: Ternary
M = 4: Quaternary

M possible messages requires M possibilities for $S(t)$:

$$\{s_1(t), s_2(t), \ldots, s_M(t)\}$$

Analysis of Digital Modem

$\{s_1(t), s_2(t), \dots, s_M(t)\}$

Waveform Channel: $R(t) = S(t) + N(t)$

Find orthonormal basis (possibly by GSOP):
$\{\phi_1(t), \phi_2(t), \dots, \phi_K(t)\}$

Ex:
$K = 1$: PAM, ASK
$K = 2$: PSK, QAM

$$\begin{pmatrix} \langle R(t), \phi_1(t) \rangle \\ \langle R(t), \phi_2(t) \rangle \\ \vdots \\ \langle R(t), \phi_K(t) \rangle \end{pmatrix} \quad \begin{pmatrix} \langle S(t), \phi_1(t) \rangle \\ \langle S(t), \phi_2(t) \rangle \\ \vdots \\ \langle S(t), \phi_K(t) \rangle \end{pmatrix} \quad \begin{pmatrix} \langle N(t), \phi_1(t) \rangle \\ \langle N(t), \phi_2(t) \rangle \\ \vdots \\ \langle N(t), \phi_K(t) \rangle \end{pmatrix}$$

Vector Channel: $\vec{\mathbf{R}} = \vec{\mathbf{S}} + \vec{\mathbf{N}}$

$\in \left\{ \vec{\mathbf{s}}^{(1)}, \vec{\mathbf{s}}^{(2)}, \dots, \vec{\mathbf{s}}^{(M)} \right\}$

can be visualized in the form of signal constellation

$\vec{\mathbf{R}}$, $\vec{\mathbf{S}}$ and $\vec{\mathbf{N}}$ are all random vectors.
Each vector contain K random variables.

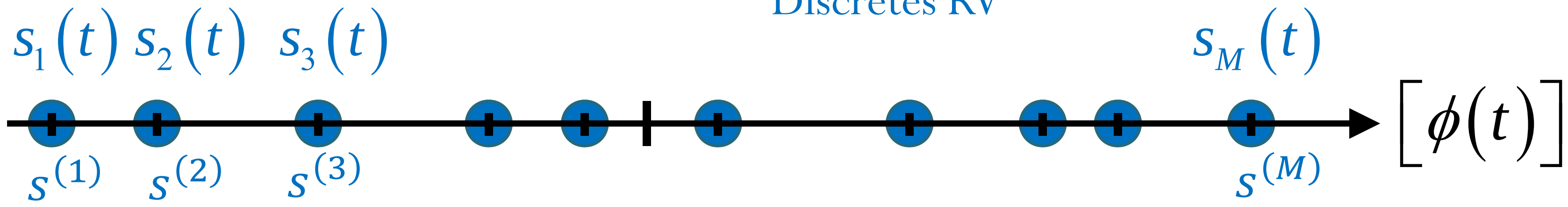
Analysis of Digital Modem: 1D

$K = 1$

$$R = S + N$$

Continuous RV (N)

Discretes RV (S)



Prior Probabilities: $p_j = P\left[S(t) = s_j(t)\right] = P\left[S = s^{(j)}\right] \equiv p_S\left(s^{(j)}\right)$

pmf of the "message"

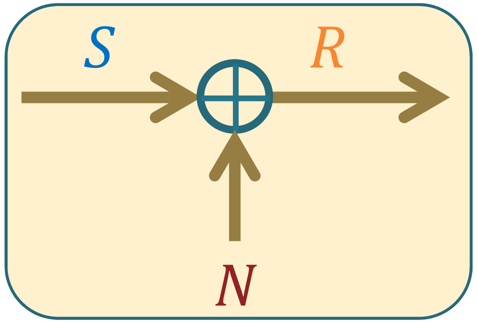
Energy: $E_j = \left\|s_j(t)\right\|^2 = \left\langle s_j(t), s_j(t)\right\rangle = \left|s^{(j)}\right|^2$

Average Energy (per Symbol): $E_s = \sum_{j=1}^{M} p_j E_j$

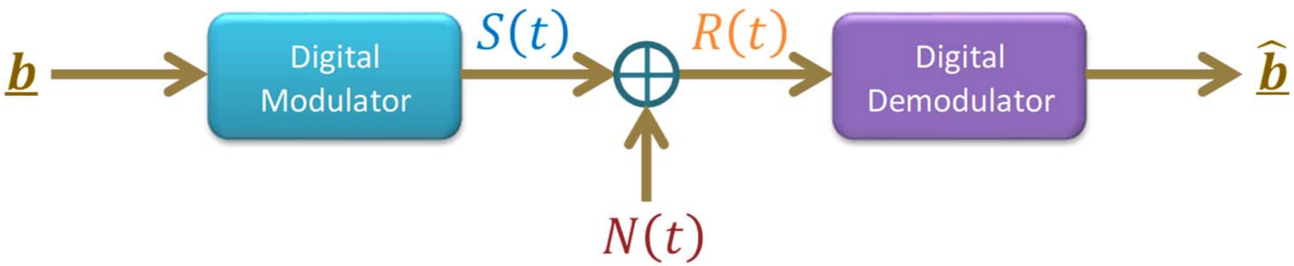
Average Energy (per Bit): $E_b = \dfrac{1}{\log_2 M} E_s$

Analysis of Digital Modem: MAP

- Model: $R = S + N$
  - We know the pmf $p_S(s)$ of S and the pdf $f_N(n)$ of N.
  - We assume that S and N are independent.



- **Goal**: Use the (observed) value of R to infer back to the value of S that was transmitted.
  - Note that once we recover the value of S, then we can map this back to the corresponding waveform $S(t)$, and consequently, recover the corresponding bit block $\underline{\mathbf{b}}$.



Review: MAP decoder

3.41. A recipe for finding the MAP decoder (optimal decoder) and its corresponding error probability:

(a) Find the **P** matrix by scaling elements in each row of the **Q** matrix by their corresponding prior probability $p(x)$.

(b) Select (by circling) the maximum value in each column (for each value of y) in the **P** matrix.

- If there are multiple max values in a column, select one. This won't affect the optimality of your answer.

(i) The corresponding x value is the value of $\hat{x}$ for that y.

(ii) The sum of the selected values from the **P** matrix is $P(\mathcal{C})$.

(c) $P(\mathcal{E}) = 1 - P(\mathcal{C})$.

Review: MAP decoder

Example 3.43. Find the MAP decoder and its corresponding error probability for the DMC channel whose $\mathbf{Q}$ matrix is given by

$$\mathbf{Q} = \begin{array}{c c} x \backslash y & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.4 & 0.3 \end{bmatrix} \end{array} \xrightarrow[\times 0.4]{\times 0.6} \begin{array}{c c} & \begin{array}{ccc} \hat{x}: 0 & 1 & 0 \end{array} \\ x \backslash y & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0.30 & 0.12 & 0.18 \\ 0.12 & 0.16 & 0.12 \end{bmatrix} \end{array} = \mathbf{P}$$

y	$\hat{x}_{MAP}(y)$
1	0
2	1
3	0

and $\underline{\mathbf{p}} = [0.6, 0.4]$. Note that the DMC is the same as in Example 3.26 but the input probabilities are different.

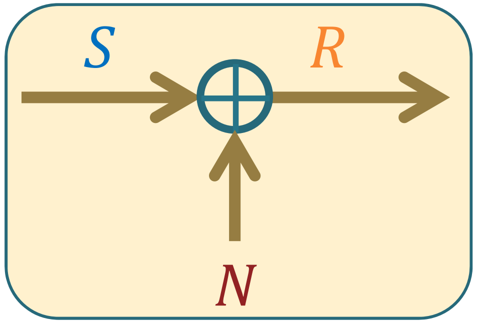
$$\hat{x}_{MAP}(y) = \begin{cases} 0, & y = 1, 3, \\ 1, & y = 2. \end{cases}$$

$$P(\mathcal{C}) = 0.3 + 0.16 + 0.18 = 0.64$$

$$P(\mathcal{E}) = 1 - 0.64 = 0.36$$

Analysis of Digital Modem: MAP

- Model: $R = S + N$
  - We know the pmf $p_S(s)$ of S and the pdf $f_N(n)$ of N.
  - We assume that S and N are independent.



- Suppose, at the receiver, we get $R = r$.
- Optimal (MAP) Detector:

$$\hat{s}_{\mathrm{MAP}}(r) = \underset{s \in \{s^{(1)}, s^{(2)}, \ldots, s^{(M)}\}}{\arg\max} \; p_S(s) f_N(r - s)$$

8.8. Graphically, here are the steps to find the MAP detector:

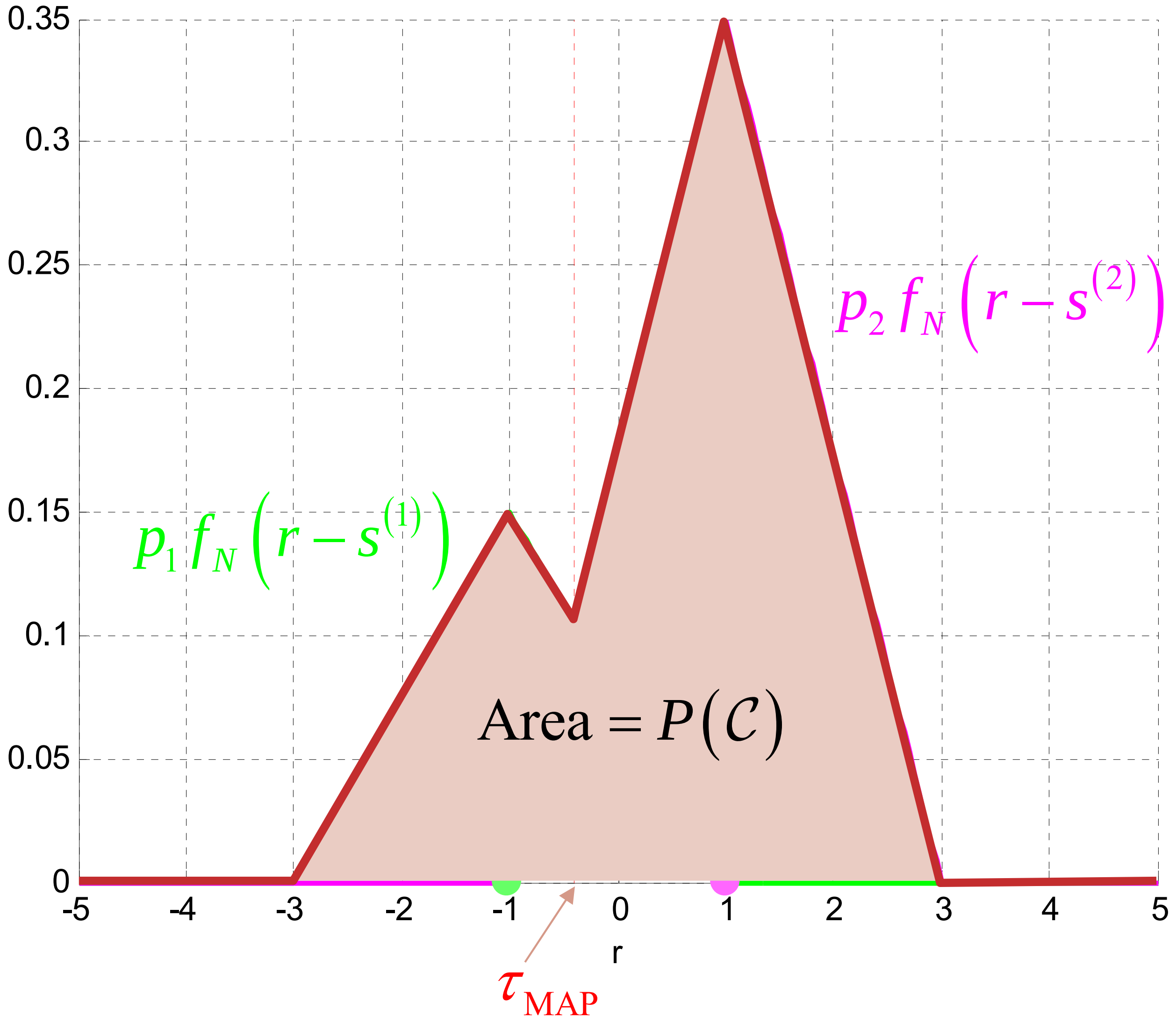
(a) Plot $p_1 f_N(r - s^{(1)})$, $p_2 f_N(r - s^{(2)})$, $\ldots$, $p_M f_N(r - s^{(M)})$.

(b) Select the maximum plot for each (observed) r value.

- If there are multiple max values, select any.
- The corresponding $s^{(j)}$ is the value of $\hat{s}_{\mathrm{MAP}}$ at r.

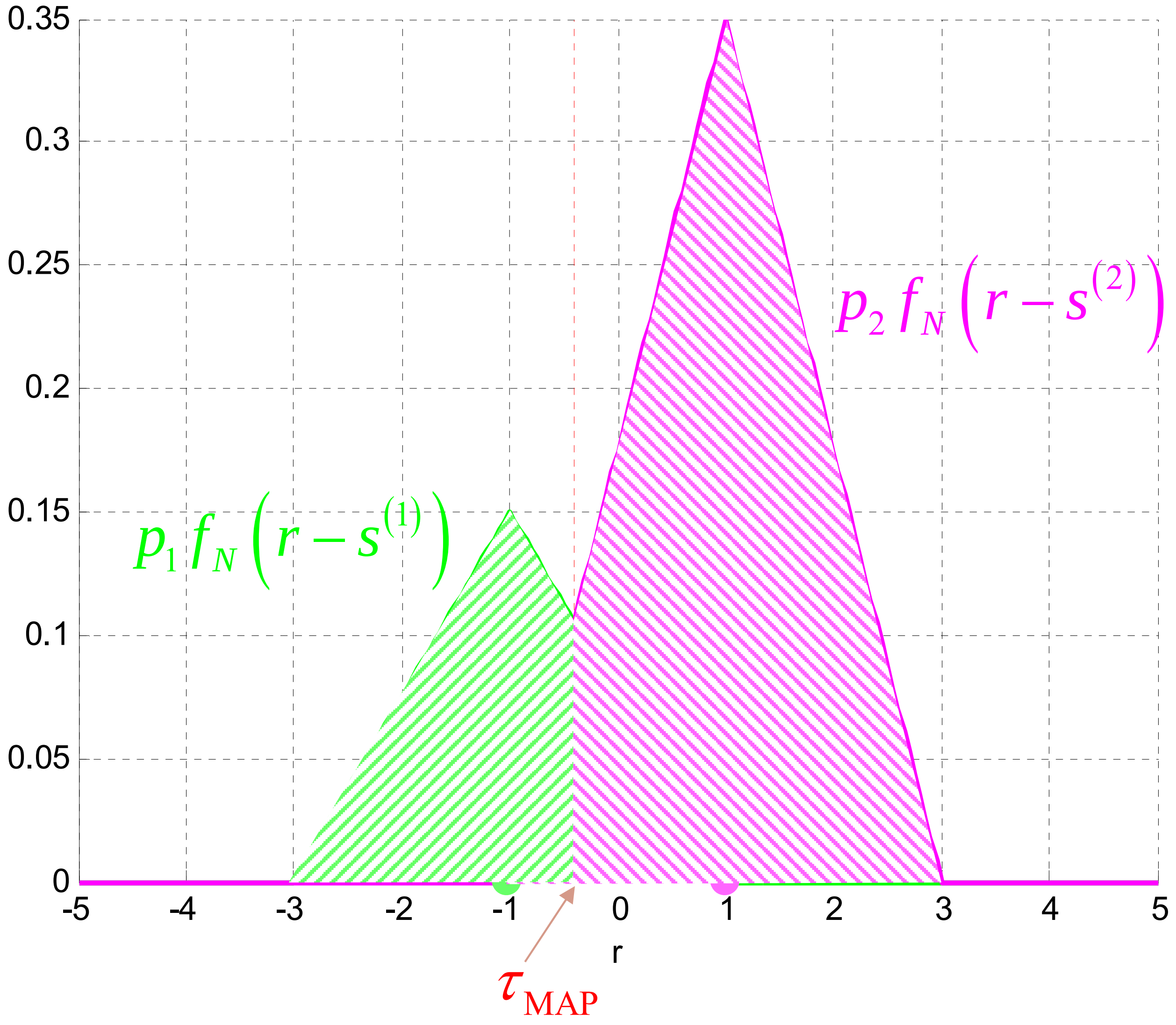
Error Probability

Ex. Binary PAM under "Triangular" Noise



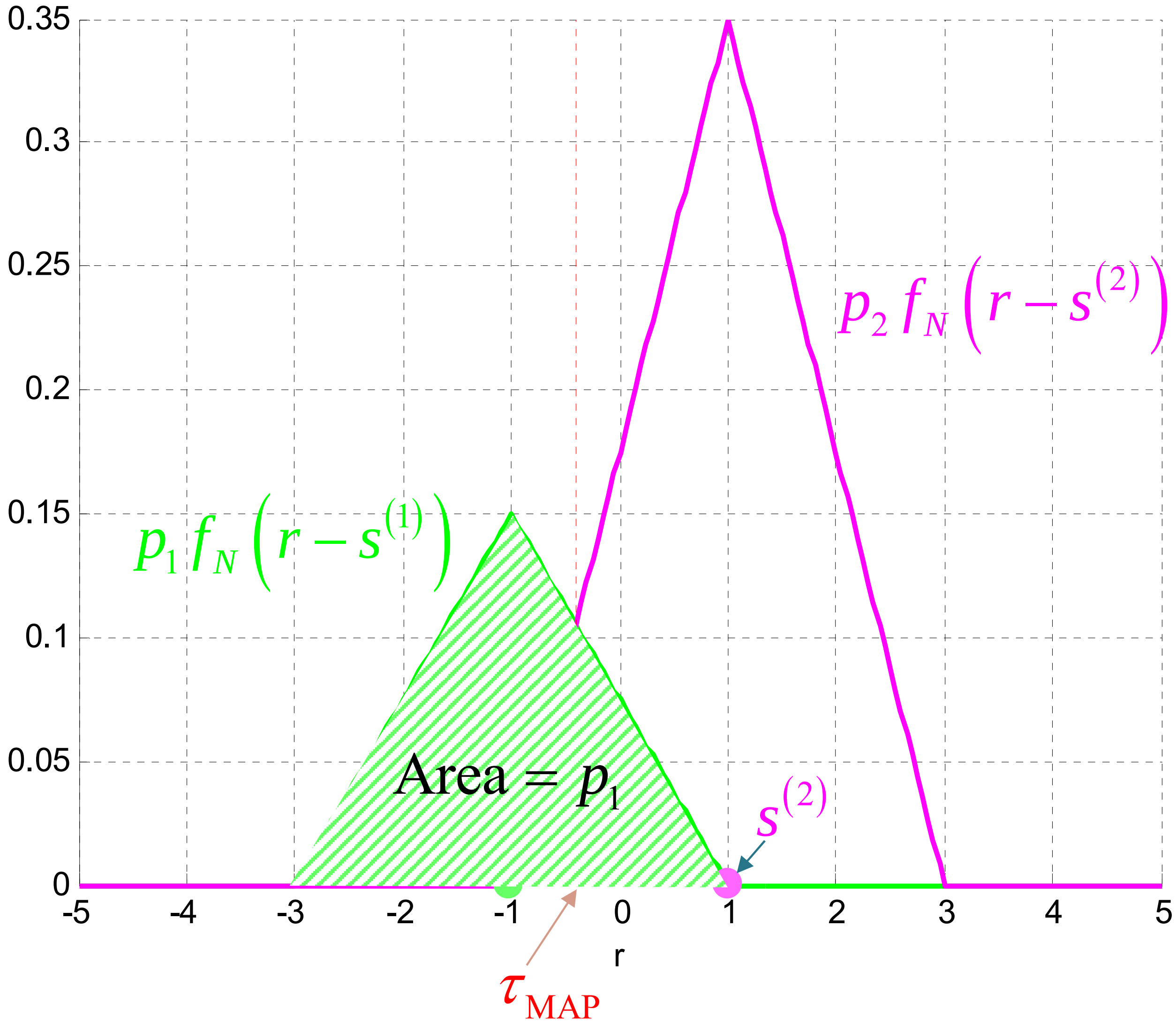
Error Probability

Ex. Binary PAM under "Triangular" Noise



Error Probability

Ex. Binary PAM under "Triangular" Noise



Error Probability

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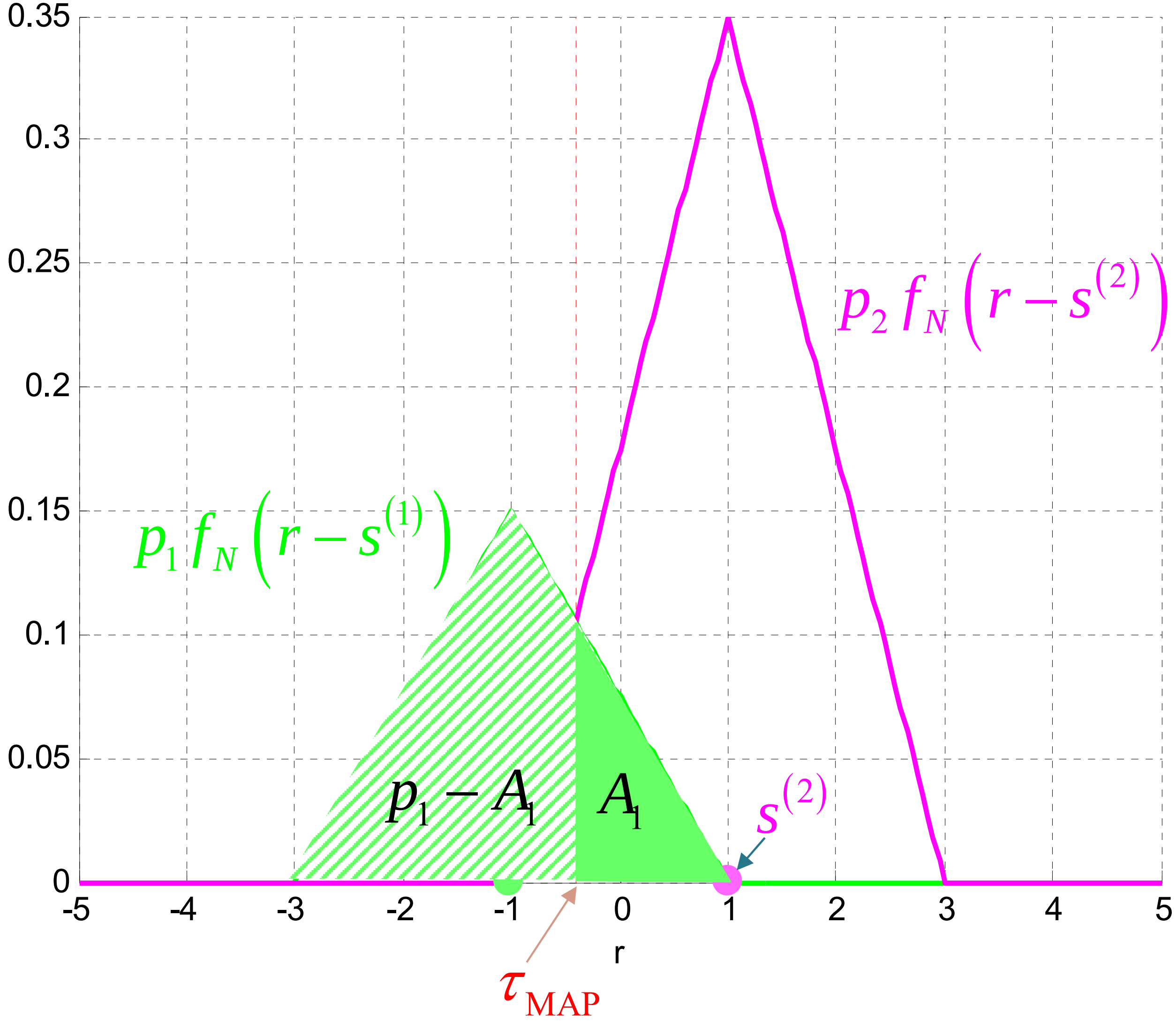

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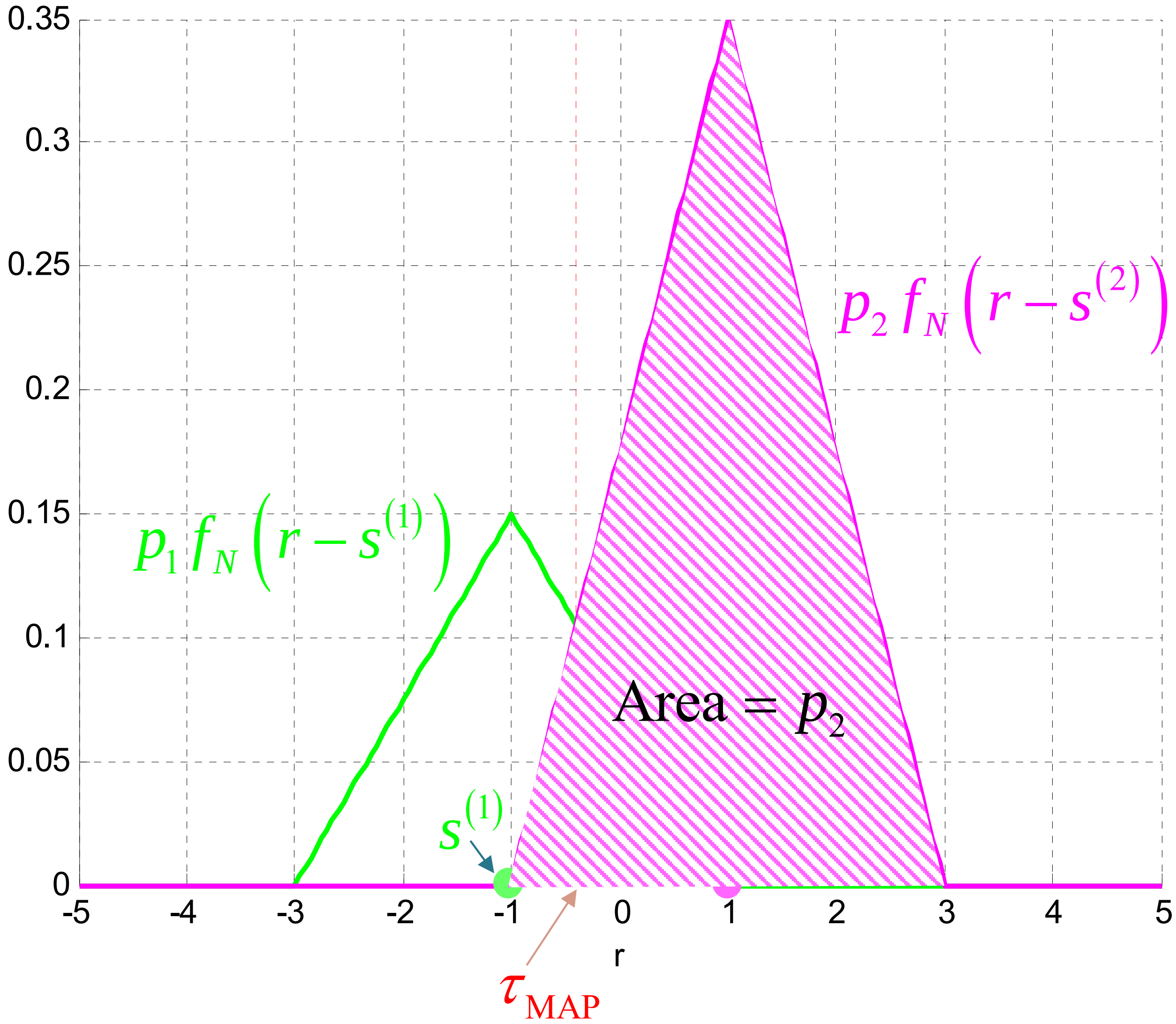

Error Probability

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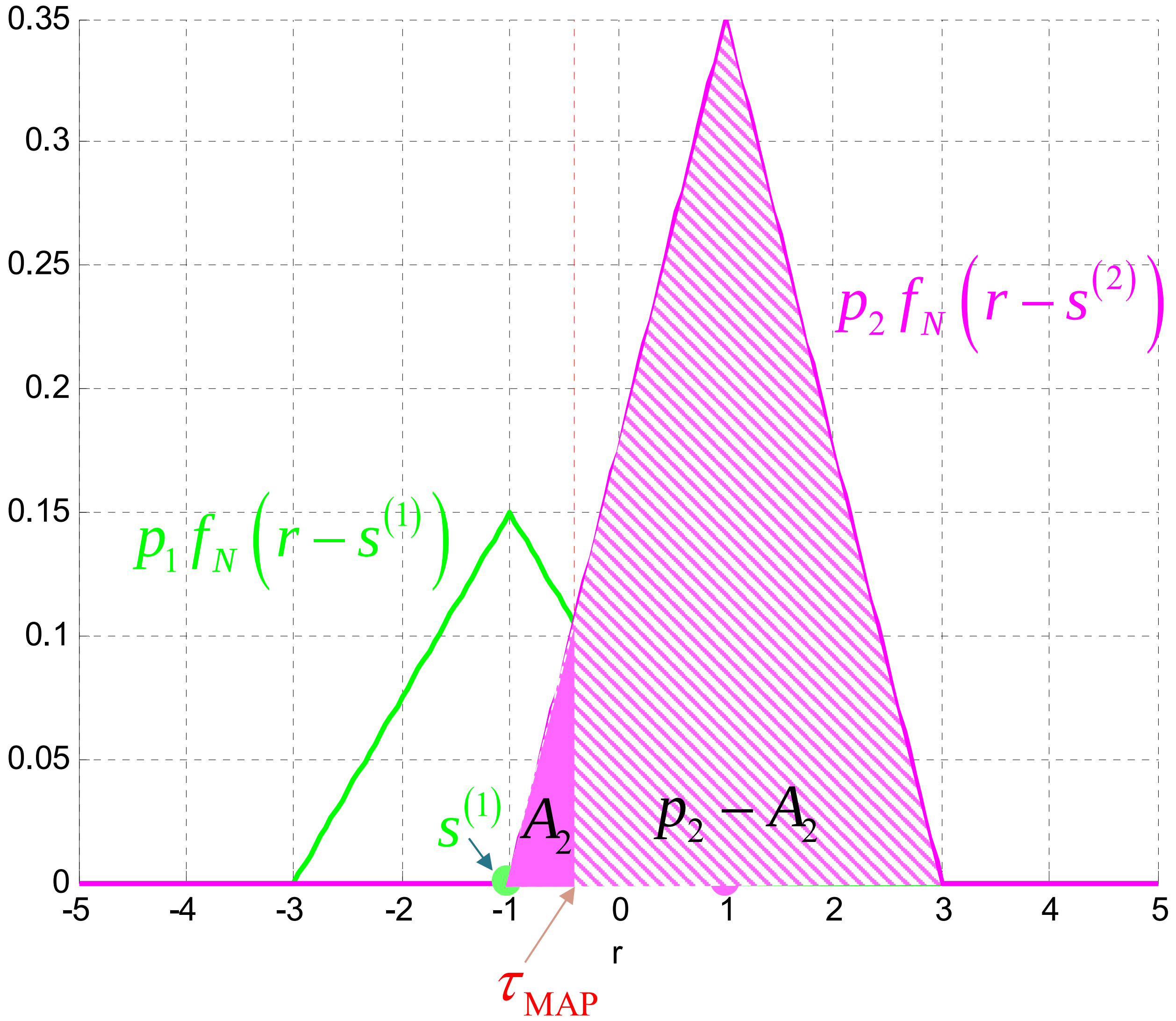

Error Probability

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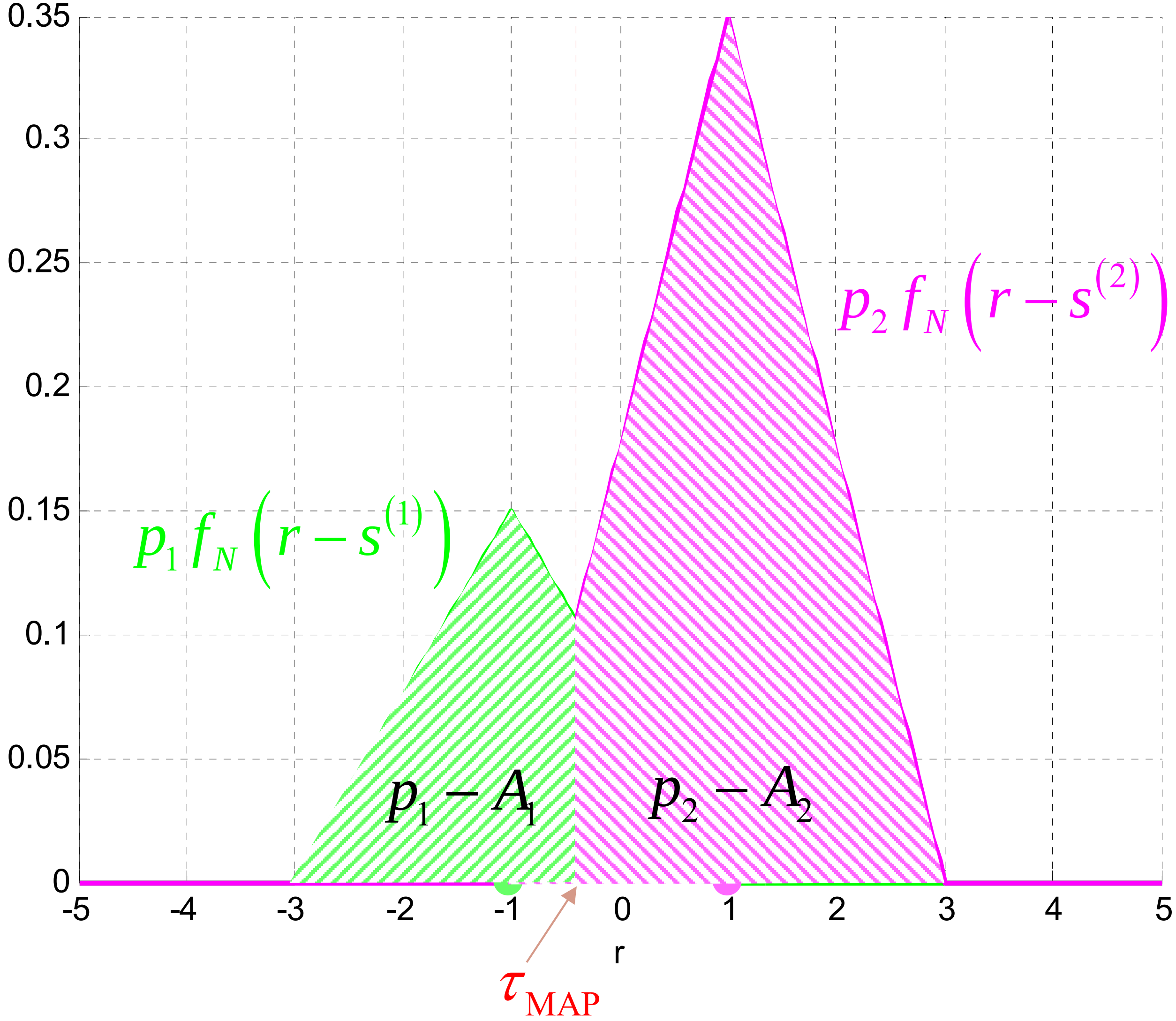
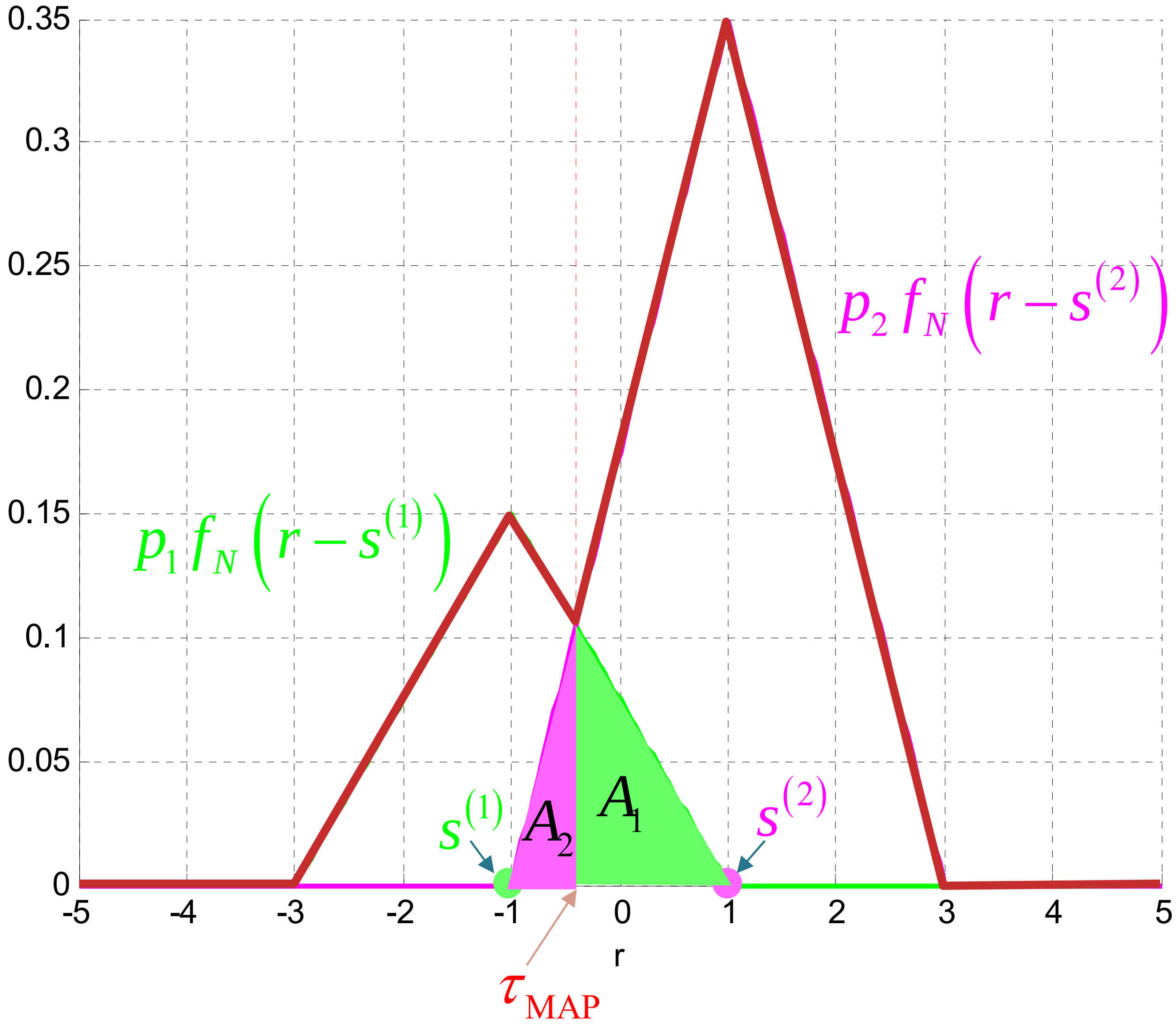


$$P(\mathcal{C}) = (p_1 - A_1) + (p_2 - A_2)$$
$$= 1 - (A_1 + A_2)$$
$$P(\mathcal{E}) = 1 - P(\mathcal{C})$$
$$= A_1 + A_2$$

Error Probability

Ex. Binary PAM under "Triangular" Noise



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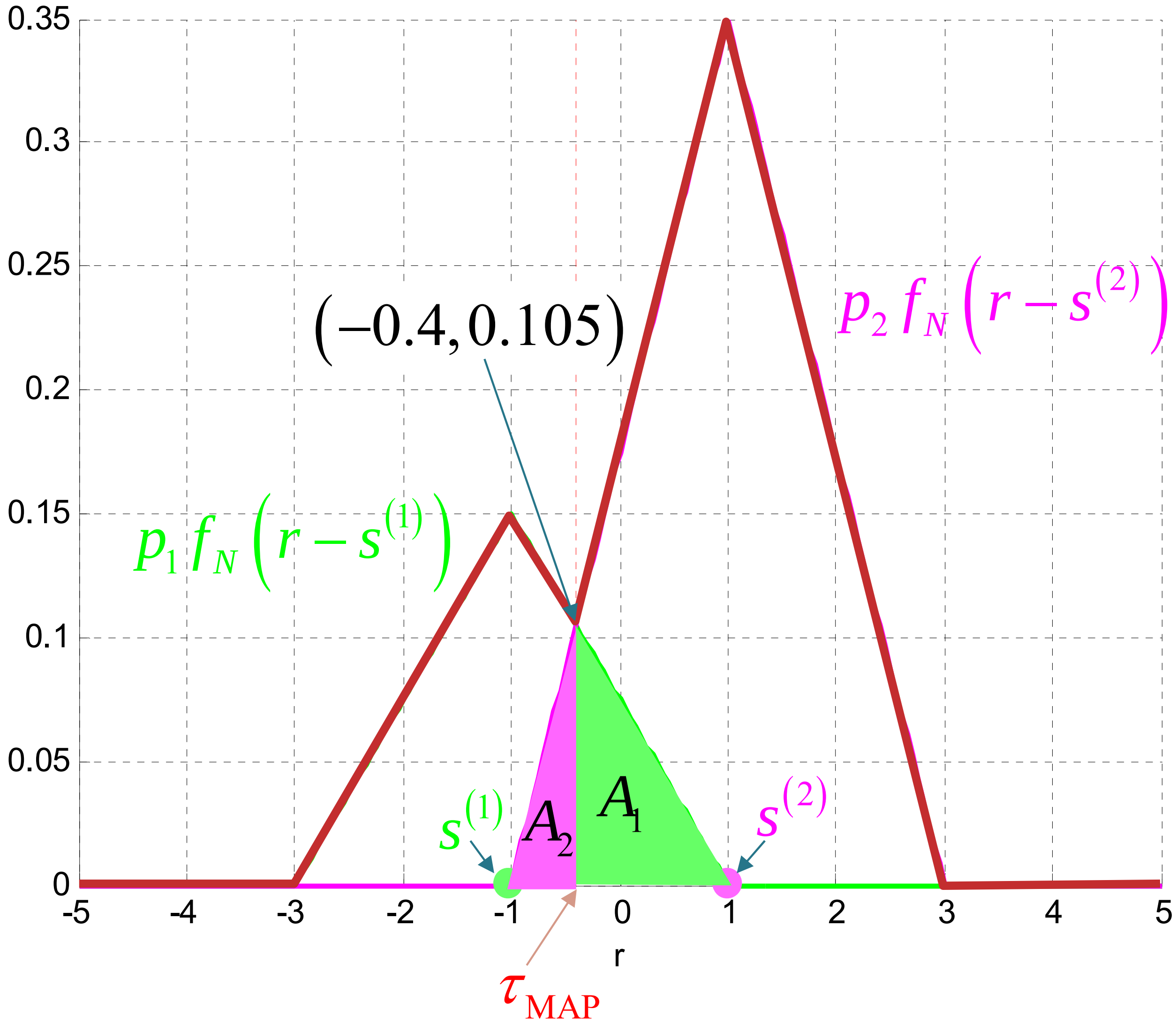
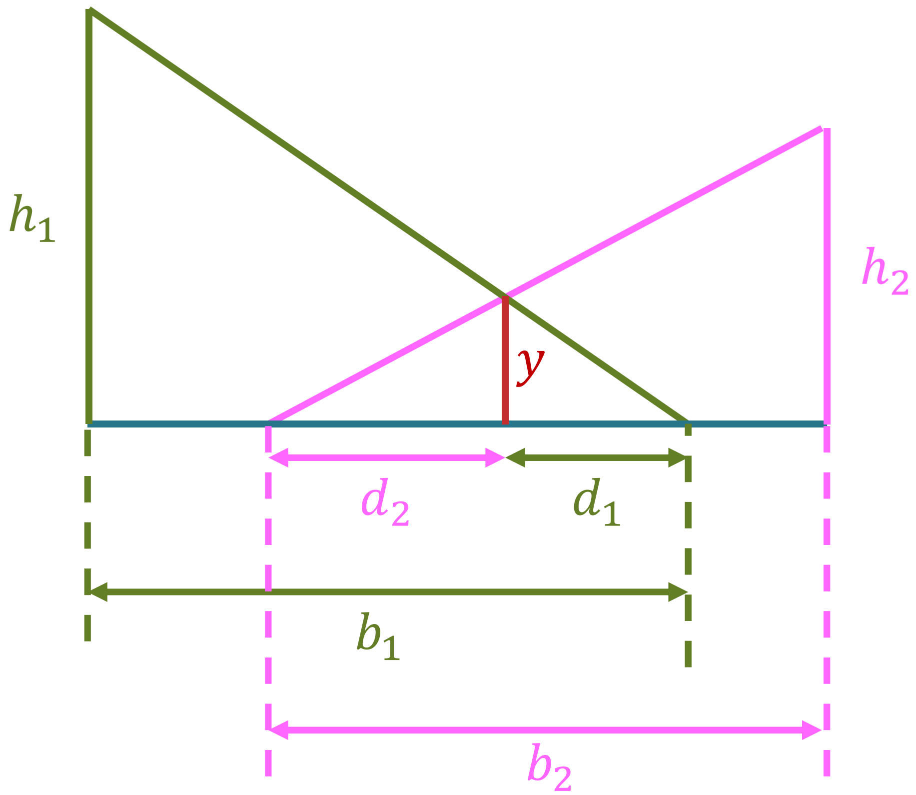


$$P(\mathcal{C}) = (p_1 - A_1) + (p_2 - A_2)$$
$$= 1 - (A_1 + A_2)$$
$$P(\mathcal{E}) = 1 - P(\mathcal{C})$$
$$= A_1 + A_2$$
$$= \frac{1}{2} \times 2 \times 0.105$$
$$= 0.105$$

Similar Triangles



Similar Triangles

- $\frac{h_1}{b_1} = \frac{y}{d_1}$
- $\frac{h_2}{b_2} = \frac{y}{d_2}$